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Large deflections of non-linear elastic rectangular plates

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Introduction

We consider a thin rectangular plate of a non-linear elastic material, hinged at the edges and loaded by a uniformly distributed transverse load p per unit area. The following investigation gives an approximate solution to the large, or membrane, deflection problem for the plate, neglecting the bending stresses.

The plate has sides $2\,a$ and $2\,b$ and thickness h. We use a coordinate system with origin at the midpoint and the xy-plane in the middle plane of the plate. The x-axis is parallel to the side of length $2\,a$, and the z-axis is perpendicular to the plate and positive in a direction opposite to the load p. The displacement has the components u, v and w in the coordinate directions. The plate has a stress-strain relation having the same analytical expression as Norton's creep law, namely

$$\varepsilon = k\,\sigma^n$$

or generalised to three dimensions [1]

$$\varepsilon_{ij} = \tfrac{3}{2}\,F(\sigma_e)\cdot s_{ij} \tag{1}$$

with

$$F(\sigma_e) = k\,\sigma_e^{n-1}$$

If equation (1) is solved for the stresses, we obtain

$$s_{ij} = \tfrac{2}{3}\,f(\varepsilon_e)\,\varepsilon_{ij} \tag{2}$$

with

$$f(\varepsilon_e) = \frac{1}{\sqrt[n]{k}}\,\varepsilon_e^{\frac{1-n}{n}}$$

The stress invariant σ_e and the strain invariant ε_e are

$$\sigma_e^2 = \tfrac{3}{2}\,s_{ij}\,s_{ij}$$

$$\varepsilon_e^2 = \tfrac{2}{3}\,\varepsilon_{ij}\cdot\varepsilon_{ij}$$

A plane stress distribution is assumed and the stress tensor, the stress deviation and the stress invariant then are

$$\sigma_{ij}=\begin{pmatrix}\sigma_x & \sigma_{xy} & 0\\ \sigma_{xy} & \sigma_y & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$s_{ij}=\begin{pmatrix}\frac{1}{3}(2\sigma_x-\sigma_y) & \sigma_{xy} & 0\\ \sigma_{xy} & \frac{1}{3}(2\sigma_y-\sigma_x) & 0\\ 0 & 0 & -\frac{1}{3}(\sigma_x+\sigma_y)\end{pmatrix} \tag{3}$$

$$\sigma_e^2=\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\sigma_{xy}^2$$

The plate is assumed incompressible ($\nu=1/2$) so we have $\varepsilon_x+\varepsilon_y+\varepsilon_z=0$. With the notation $\varepsilon_{xy}=1/2\,\gamma_{xy}$, where γ_{xy} is the usual shearing strain, and neglecting γ_{xz} and γ_{yz}, the strain tensor and the strain invariant are

$$\varepsilon_{ij}=\begin{pmatrix}\varepsilon_x & \varepsilon_{xy} & 0\\ \varepsilon_{xy} & \varepsilon_y & 0\\ 0 & 0 & -(\varepsilon_x+\varepsilon_y)\end{pmatrix}$$

$$\varepsilon_e^2=\frac{4}{3}(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_x\varepsilon_y+\varepsilon_{xy}^2) \tag{4}$$

Only deflections larger than the plate thickness are considered, and the bending stresses can be neglected when compared with the membrane stresses. The relationship between strain and displacement then is (reference [2], p, 304.)

$$\left.\begin{aligned}\varepsilon_x&=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y&=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \varepsilon_{xy}&=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\} \tag{5}$$

Energy method solution

For the rectangular membrane the equations of equilibrium together with a relationship between stress and displacement are very complicated, but it is possible to determine an approximate solution to the membrane problem using an energy method. A solution is assumed with a number of arbitrary constants, which are then determined so that the potential energy is a minimum. The general expression for the potential energy for the transverse loaded plate is

$$U=\iiint[\sigma_x d\varepsilon_x+\sigma_y d\varepsilon_y+2\sigma_{xy}d\varepsilon_{xy}]\,dx\,dy\,dz+\iint p\,w\,dx\,dy-\int\sum(P\delta+M\theta)\,ds \tag{6}$$

The first term is the strain energy, the second term is the work of outer forces and the third term is the work of the reactions. The third term is zero for a simply supported or a clamped plate, and is omitted. The strain energy per unit volume can be transformed by (2), (3) and (4) and integrated. The z-integration in (6) can also be carried out, and then the potential energy becomes

$$U = K \iint \varepsilon_e^{2\varkappa}\, d\,x\, d\,y + \iint p\, w\, d\,x\, d\,y \tag{7}$$

where

$$K = \frac{n}{n+1} \cdot \frac{h}{\sqrt[n]{k}}$$

$$\varkappa = \frac{n+1}{2\,n}$$

The integration is carried out over the plate area.

For the simply supported plate we assume the solution

$$\left.\begin{aligned} w &= A\,(x^2 - a^2)\,(y^2 - b^2) \\ u &= B\,x\,(x^2 - a^2)\,(y^2 - b^2) \\ v &= C\,y\,(x^2 - a^2)\,(y^2 - b^2) \end{aligned}\right\} \tag{8}$$

From (4), (5) and (8) we obtain

$$\begin{aligned} \varepsilon_e^2 = {} & \tfrac{16}{3} A^4 [x^2 (y^2 - b^2)^2 + y^2 (x^2 - a^2)^2]^2 + \\ & + \tfrac{16}{3} A^2 B [x^2 (3\,x^2 - a^2)(y^2 - b^2)^2 + \tfrac{1}{2} y^2 (5\,x^2 - a^2)(x^2 - a^2)^2] (y^2 - b^2) + \\ & + \tfrac{16}{3} A^2 C [y^2 (3\,y^2 - b^2)(x^2 - a^2)^2 + \tfrac{1}{2} x^2 (5\,y^2 - b^2)(y^2 - b^2)^2] (x^2 - a^2) + \\ & + \tfrac{4}{3} B^2 [(3\,x^2 - a^2)^2 (y^2 - b^2)^2 + x^2 y^2 (x^2 - a^2)^2] + \\ & + \tfrac{4}{3} C^2 [(3\,y^2 - b^2)^2 (x^2 - a^2)^2 + x^2 y^2 (y^2 - b^2)^2] + \\ & + \tfrac{4}{3} B\,C [(3\,x^2 - a^2)(3\,y^2 - b^2) + 2\,x^2 y^2] (x^2 - a^2)(y^2 - b^2) \end{aligned} \tag{9}$$

It is not possible to integrate (7) for $\varkappa \neq 1$. In formulating the minimum conditions

$$\frac{\partial U}{\partial A} = 0; \ \frac{\partial U}{\partial B} = 0; \ \frac{\partial U}{\partial C} \tag{10}$$

we have to change the order of differentiation and integration, and after some rearrangement these conditions can be written

$$\left.\begin{aligned} & \int_{-a}^{a} \int_{-b}^{b} \frac{\frac{\partial}{\partial A}(\varepsilon_e^2)}{\varepsilon_e^{2(1-\varkappa)}}\, d\,x\, d\,y = -\frac{32}{9} \frac{p\,a^3 b^3 \sqrt[n]{k}}{h} \\ & \int_{-a}^{a} \int_{-b}^{b} \frac{\frac{\partial}{\partial B}(\varepsilon_e^2)}{\varepsilon_e^{2(1-\varkappa)}}\, d\,x\, d\,y = 0 \\ & \int_{-a}^{a} \int_{-b}^{b} \frac{\frac{\partial}{\partial C}(\varepsilon_e^2)}{\varepsilon_e^{2(1-\varkappa)}}\, d\,x\, d\,y = 0 \end{aligned}\right\} \tag{11}$$

From a dimensional analysis of (8), (9) and (11) we find that it is possible to express the constants A, B and C as

$$\left.\begin{aligned} A &= -\alpha\frac{1}{a^3}\left(\frac{p\,a\sqrt[n]{k}}{h}\right)^{\frac{n}{n+2}} \\ B &= \beta\,a^2 A^2 \\ C &= \gamma\,a^2 A^2 \end{aligned}\right\} \tag{12}$$

where the coefficients α, β and γ depend only on the ratio $b/a=\lambda$. We define a polynomial H as

$$H = 4F_1 + 4\beta a^2 F_2 + 4\gamma a^2 F_3 + \beta^2 a^4 F_4 + \gamma^2 a^4 F_5 + \beta\gamma a^4 F_6 \tag{13}$$

so that

$$\varepsilon_e^2 = \tfrac{4}{3} A^4 H \tag{14}$$

The polynomials F are obtained from (9), (13) and (14). With (12), (13) and (14) we can rewrite (11) and get

$$\left.\begin{aligned} &\left(\frac{\alpha}{a^3}\right)^{\frac{n+2}{n}} \int_0^a\!\!\int_0^b \frac{2F_1+\beta a^2 F_2+\gamma a^2 F_3}{H^{1-\varkappa}}\,dx\,dy = \frac{1}{12}\left(\frac{4}{3}\right)^{1-\varkappa} a^2 b^3 \\ &\int_0^a\!\!\int_0^b \frac{4F_2+2\beta a^2 F_4+\gamma a^2 F_6}{H^{1-\varkappa}}\,dx\,dy = 0 \\ &\int_0^a\!\!\int_0^b \frac{4F_3+2\gamma a^2 F_5+\beta a^2 F_6}{H^{1-\varkappa}}\,dx\,dy = 0 \end{aligned}\right\} \tag{15}$$

The three unknowns α, β and γ can now be calculated from (15) with a digital computer. The results of such calculations are given in table 1. The maximum deflection of the plate is

$$w_0 = -\alpha\lambda^2 a\left(\frac{p\,a\sqrt[n]{k}}{h}\right)^{\frac{n}{n+2}} \tag{16}$$

The linear case

In the special case $n=1$ ($\varkappa=1$) we can integrate (7) exactly and get

$$U = K\,(P_1 A^4 + P_2 A^2 B + P_3 A^2 C + P_4 B^2 + P_5 C^2 + P_6 B C) + \tfrac{16}{9} p a^3 b^3 A$$

where P_1 to P_6 are

$$P_1 = \frac{64\cdot 128}{21\cdot 25\cdot 63}\,(7a^4 + a^2 b^2 + 7b^4)\,a^5 b^5$$

Table 1.

$b/a=\lambda$	n	β	γ	α	$\alpha\lambda^2$
1	1	0.442	0.442	0.694	0.694
	3	0.446	0.446	0.494	0.494
	5	0.449	0.449	0.426	0.426
	7	0.451	0.451	0.392	0.392
	10	0.452	0.452	0.365	0.365
	∞	0.456	0.456	0.293	0.293
1.5	1	1.172	0.280	0.375	0.844
	3	1.026	0.419	0.280	0.630
	5	0.963	0.463	0.246	0.554
	7	0.928	0.485	0.229	0.516
	10	0.896	0.504	0.215	0.485
	∞	0.801	0.553	0.178	0.401
2	1	2.195	0.0863	0.226	0.904
	3	1.856	0.370	0.173	0.691
	5	1.684	0.450	0.153	0.613
	7	1.578	0.490	0.144	0.574
	10	1.481	0.523	0.136	0.542
	∞	1.151	0.613	0.114	0.457
3	1	5.102	−0.322	0.105	0.944
	∞	1.858	0.684	0.0559	0.503
∞	1				0.96

$$P_2=\frac{56\cdot 128}{21\cdot 25\cdot 63}(a^2-12\,b^2)\,a^5\,b^5$$

$$P_3=\frac{56\cdot 128}{21\cdot 25\cdot 63}(b^2-12\,a^2)\,a^5\,b^5$$

$$P_4=\frac{7\cdot 128}{21\cdot 25\cdot 63}(5\,a^2+84\,b^2)\,a^5\,b^3$$

$$P_5=\frac{7\cdot 128}{21\cdot 25\cdot 63}(5\,b^2+84\,a^2)\,a^3\,b^5$$

$$P_6=\frac{147\cdot 128}{21\cdot 25\cdot 63}\,a^5\,b^5$$

The minimum conditions (10) then give

$$\left.\begin{aligned} B&=\frac{P_3P_6-2\,P_2P_5}{4\,P_4P_5-P_6^2}\,A^2=\beta\,a^2\,A^2\\ C&=\frac{P_2P_6-2\,P_3P_4}{4\,P_4P_5-P_6^2}\,A^2=\gamma\,a^2\,A^2\\ A&=-\sqrt[3]{\frac{16}{9}}\,\frac{a\,b}{\sqrt[3]{2\,P_1+\beta\,a^2\,P_2+\gamma\,a^2\,P_3}}\sqrt[3]{\frac{p\,k}{h}}=-\alpha\,\frac{1}{a^3}\sqrt[3]{\frac{p\,a\,k}{h}}\end{aligned}\right\}\qquad(17)$$

For the square elastic plate with $\nu = \frac{1}{2}$ we obtain from table 1 or from (17) $\alpha = 0.694$. For $\nu = 1/4$ we can find the value $\alpha = 0.802$ in [2] p. 346. From (17) we can determine the maximum deflection for a very long plate ($b \gg a$). In this case the quantity $\alpha \lambda^2$ in (16) will be equal to 0.96.

Simplified solution

For the non-linear case ($\varkappa \neq 1$) we have to choose a simpler solution than (8) to be able to evaluate it without a computer. We obtain such a solution by putting $B = C = 0$ in (8), and the energy expression then becomes

$$U = 4K\left(\tfrac{16}{3}\right)^{\varkappa} A^{4\varkappa} \cdot I + \tfrac{16}{9}\, p\, a^3 b^3 A$$

with

$$I = \int_0^a \int_0^b \left[x^2 (y^2 - b^2)^2 + y^2 (x^2 - a^2)^2\right]^{2\varkappa} dx\, dy$$

The integral I must be evaluated numerically, but this can easily be done by hand-calculation. For the constant A we get

$$A = -\left[\frac{n+1}{n}\left(\frac{3}{16}\right)^{\varkappa} \frac{p\, a^3 b^3 \sqrt{k}}{9 \varkappa h I}\right]^{\frac{n}{n+2}} = -\alpha \frac{1}{a^3}\left(\frac{p\, a \sqrt{k}}{h}\right)^{\frac{n}{n+2}}$$

In the case of a square plate this simpler solution gives a maximum deflection which is about 10 % less than the one obtained by the complete solution (8).

Discussion of the accuracy

To get an impression of the accuracy of the method used above, we can apply it to the corresponding problem for the circular plate, where an exact solution is determined by Odqvist [3]. For the circular plate we assume

$$w = A(r^2 - a^2)$$
$$u = B r (r^2 - a^2)$$

where r is the radius vector, a the radius of the plate and u and w the radial and transverse displacement. For $n = 1$, which case is easy to calculate, the energy method gives a maximum deflection, which is 1 or 2 % too large. A simpler solution with $B = 0$ gives a maximum deflection which is between 10 and 14 % too small. It is reasonable to assume that the deviation from the exact solution is somewhat larger for a square or rectangular plate than for the circular plate. The accuracy is also less for longer plates. Other examples of the accuracy of the energy method can be obtained from small deflection problems with $n = 1$, for instance from the problem of a uniformly loaded clamped rectangular plate, where a comparable energy method [4] gives about 5 % error for $b/a = 1$, 12 % error for $b/a = 2$ and about 25 % error, when b/a becomes very large. All these values are above the exact value.

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